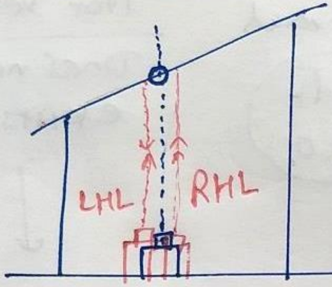
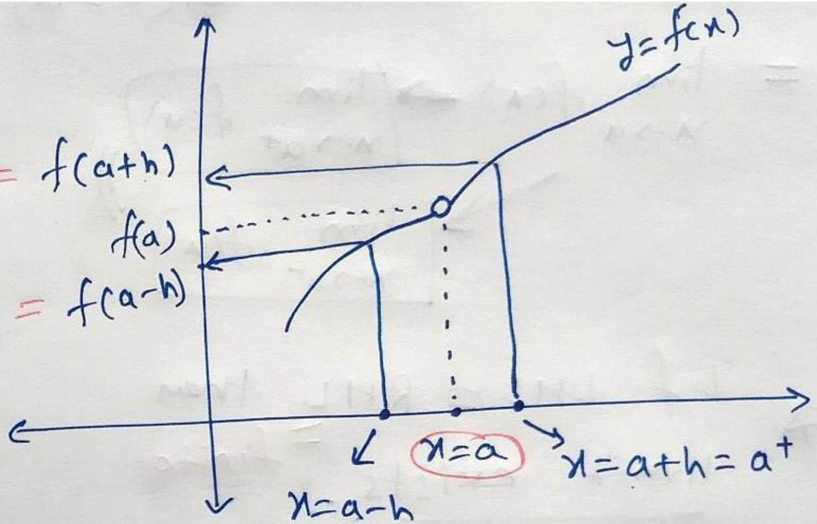
# LIMITS & DERIVATIVES

## Limit:




machine.

$d = S \times t$




$h \approx 0$ $\quad h = 0.000000$

$h \to 0$

**Definition.** :→ limit of a function $y = f(x)$ at $x = a$ is the value of function in the neighborhood of $x = a$.

RHL = Right Hand limit $= \lim_{x \to a^+} f(x)$

$= \lim_{x \to a+h} f(x) = f(a^+) = \lim_{h \to 0} f(a+h)$

LHL = Left Hand limit $= \lim_{x \to a^-} f(x)$

$= \lim_{x \to a-h} f(x) = f(a^-) = \lim_{h \to 0} f(a-h)$

★ limit of a function at $(x=a)$

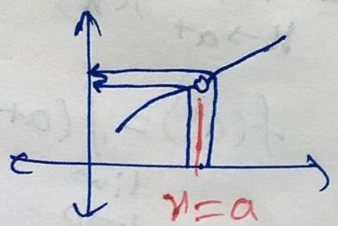
$$= \lim_{x\to a} f(x) \Rightarrow \boxed{\lim_{x\to a^+} f(x)} = \boxed{\lim_{x\to a^-} f(x)}$$

★ If LHL = RHL then $=$ finite

limit exists.

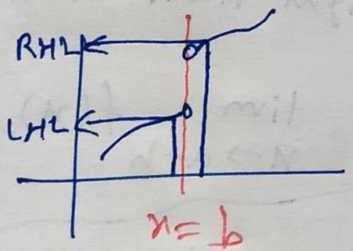
★ If LHL ≠ RHL then

limit does not exist.




LHL = RHL

Limit exists




LHL ≠ RHL

limit Does not exist

Values $\rightarrow f(x) = 0,$

| Normal | Not Defined ($\infty$) | Indeterminate Forms |
|---|---|---|
| $5, \sqrt{2}, 3, 0,$ | Does not exists $\sqrt{-1}=i$ | |
| OK | OK | Dangerous |

★ Chapter-13 Limit

Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty\cdot 0, \infty-\infty, 0^0, \infty^0, 1^\infty$$

Not exact

$$\boxed{(\text{exact } 1)^\infty = 1}$$

e.g. $\frac{0}{0}$ Find the value of function $f(x) = \frac{x^2-9}{x-3}$ at $x=3$.

Domain $x-3 \neq 0$

$x \neq 3$

we can not find the value of $f(x)$ at $x=3$ but we can find value of $f(x)$ in the neighborhood of $x=3$.

→ limit at $x=3$

$f(3) = \frac{(3)^2-9}{(3)-3} = \frac{9-9}{3-3} = \frac{0}{0}$

Indeterminate form.

$f(x) = \frac{x^2-9}{x-3}$ By Factorisation

$f(x) = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$ $x=3$

$f(x) = x+3$

$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3}$ $\frac{0}{0}$

$= \lim_{x\to 3} (x+3)$

$= 3+3$

$= 6$ ✓

$RHL = 3^+ + 3 = 6^+$

$LHL = 3^- + 3 = 6^-$

## Algebra of limits

(i) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} (f(x) \times g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$

(iii) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(iv) $\lim_{x \to a} (K \cdot f(x)) \longrightarrow K \cdot \lim_{x \to a} f(x)$

(v) $\lim_{x \to a} (f(x))^{g(x)} = \left(\lim_{x \to a} f(x)\right)^{\left(\lim_{x \to a} g(x)\right)}$

## Limit for Polynomial Functions. = Continuous

$f(x) = 3x+1$

→ do not calculate limits Differently ~~LHL, RHL~~

→ For limit, Directly put the value of $x$.

e.g. $f(x) = 3x+1$

limit at $x = 2$ = ?

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} (3x+1)$$

$$= 3(2)+1 = 7$$

## Limit for Rational $F^n$. $= \frac{f(x)}{g(x)}$

$\left(\frac{0}{0}\right)$ → By Factorisation

→ By Standard Form

$$\boxed{\lim_{x\to a} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}}$$

e.g. $\frac{2x+3}{x-1}$ , $\frac{x^2+3x+5}{x^3-9}$

Proof of $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n \cdot a^{n-1}$

$$LHS = \lim_{x \to a} \frac{x^n - a^n}{x - a} \quad \text{Factorise} \quad \frac{0}{0}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}\left(x^{n-1} + x^{n-2} \cdot a^1 + x^{n-3} \cdot a^2 + \cdots + x^0 \cdot a^{n-1}\right)}{\cancel{(x-a)}}$$

$$= a^{n-1} + a^{n-2} \cdot a^1 + a^{n-3} \cdot a^2 + \cdots + a^0 \cdot a^{n-1}$$

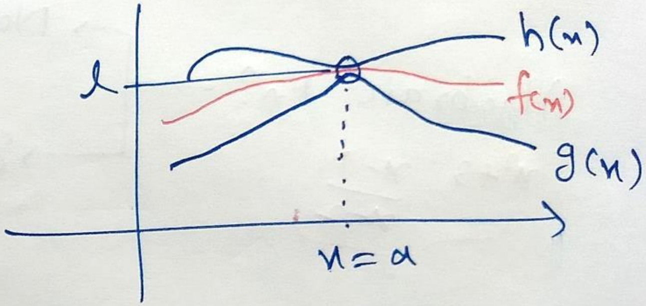
$$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ times.}}$$

$$= n \cdot a^{n-1}$$

e.g. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = n \cdot a^{n-1} = 2 \cdot 3^{2-1} = 2 \times 3^1 = 6$

$n = 2$ $a = 3$

## Sandwich Theorem :


Always

$$g(x) \le f(x) \le h(x)$$

$$\Rightarrow \lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$$

Tough

| $\lim_{x\to a} g(x)$ | $\lim_{x\to a} f(x)$ | $\lim_{x\to a} h(x)$ |
|---|---|---|
| $l$ | $l$ | $l$ |
| Easy | | Easy |

$\theta \to 0$

$\sin\theta \to \theta$

By Sandwich theorem

Important Result

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$\frac{\sin 0}{0} = \frac{0}{0}$

$$\lim_{f(x)\to 0} \frac{\sin f(x)}{f(x)} = 1$$

$$\boxed{\lim_{x\to 0} \cos x = 1}$$

Final - Conclusion:

Algebraic Fn$^n$ ($x^2+3$, $\frac{x-1}{x-2}$)
- → Normal value = Answer ✓
- → Indeterminate Form ($\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0\times\infty$)
  - → By Factorisation
  - → By Standard Form: $\lim_{x\to a}\frac{x^n-a^n}{x-a} = n.a^{n-1}$

Trigonometric Fn$^n$.
- → Normal Value ✓
- → Indeterminate Form ($\frac{0}{0}$, ...) → By Standard Form: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Piecewise Fn$^n$ → LHL & RHL →
- If LHL = RHL (= finite) → Limit exists
- If LHL ≠ RHL → Limit Does not exists.

$$f(x) = \begin{cases} x+3, & x \geq 5 \text{ RHL} \\ x-1, & x < 5 \text{ LHL} \end{cases}$$

$\lim_{x\to 5} f(x)$
- LHL = $f(5^-)$ = 4
- RHL = $f(5^+)$ = 8 #

Exercise - 12.1

Function $y = f(x)$ (limit at $x = a$) $\lim_{x \to a}$ → put $x = a$ in $y = f(x)$ (Directly) →

→ Normal No. $(2, 3, -1, \frac{1}{2}, 0, \infty ..) =$ Answer

→ Indeterminate Form $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty\right)$ → ways

- Factorisation
- Standard forms
  - $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$
  - $\lim_{x \to 0} \frac{\sin x}{x} = 1$

**Q.1** $\lim_{x \to 3} x+3$

(Directly put $x=3$)

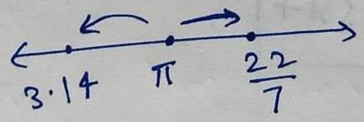
$= 3+3$

$= 6$

**Q.2** $\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$

Put $x = \pi$ Directly

$= \pi - \frac{22}{7}$




**Q.3** $\lim_{r \to 1} \pi r^2$

Put $r = 1$

$= \pi (1)^2$

$= \pi$

**Q.4** $\lim_{x \to 4} \frac{4x+3}{x-2}$

$x = 4$ put.

$= \frac{16+3}{4-2} = \frac{19}{2}$

Q.5 $\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Put $x = -1$

$= \frac{(-1)^{10} + (-1)^5 + 1}{(-1) - 1}$

$= \frac{\cancel{1} - \cancel{1} + 1}{-1-1} = \frac{1}{-2}$ ✓

Q.6 $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Put $x = 0$ $\frac{1^5 - 1}{0} = \frac{0}{0}$ = Indeterminate Form

$(x+1)^5 \rightarrow$ Expand

$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10.x^2 + 5x + 1$

$\swarrow$

$(x+1)^2 . (x+1)^3$ $^5C_0$

$\Rightarrow \lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

$= \lim_{x \to 0} \frac{(x^5 + 5.x^4 + 10.x^3 + 10.x^2 + 5.x + \cancel{1}) - \cancel{1}}{x}$

$= \lim_{x \to 0} (x^4 + 5x^3 + 10x^2 + 10x + 5)$

$= 0 + 0 + 0 + 0 + 5$ Put $x = 5$

$= 5$

Q.7 $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Put $x = 2$ Direct $\frac{12 - 2 - 10}{4 - 4} = \frac{0}{0}$ form

(Factorisation)

$= \lim_{x \to 2} \frac{3x^2 - 6x + 5x - 10}{(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{3x(x-2) + 5(x-2)}{(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(3x+5)}{\cancel{(x-2)}(x+2)}$

Put $x = 2$

$= \frac{6+5}{2+2} = \frac{11}{4}$

Q.8 $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$ $81 = 3^4$

Put $x = 3$ Directly $\frac{3^4 - 81}{2(3)^2 - 5(3) - 3} = \frac{0}{0}$ Form

$= \lim_{x \to 3} \frac{(x^2)^2 - (3^2)^2}{2x^2 - 6x + x - 3}$

$= \lim_{x \to 3} \frac{(x^2 - 3^2)(x^2 + 3^2)}{2x(x-3) + 1(x-3)}$

$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)(x^2+9)}{\cancel{(x-3)}(2x+1)}$

$= \lim_{x \to 3} \frac{(x+3)(x^2+9)}{2x+1}$

$= \frac{6 \times 18}{7} = \frac{108}{7}$

Q.9 $\lim_{x\to 0} \frac{ax+b}{cx+1}$

Put $x=0$

$= \frac{0+b}{0+1} = b$ ✓

Q.10 $\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Put $z=1$ $\frac{1-1}{1-1} = \frac{0}{0}$ Form

Standard Form

$$\boxed{\lim_{x\to a} \frac{x^n-a^n}{x-a} = n.a^{n-1}}$$

$\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

$(z-1)$ multiply & Divide

$= \lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} \times \frac{z-1}{z-1}$

$= \lim_{z\to 1} \frac{z^{1/3}-1^{1/3}}{z-1} \times \left(\frac{z-1}{z^{1/6}-1}\right)$

$n.a^{n-1}$ $\quad \frac{1}{n.a^{n-1}}$

$= \left[\frac{1}{3}.(1)^{1/3-1}\right] \times \left(\frac{1}{\frac{1}{6}.1^{1/6-1}}\right)$

$= \frac{1}{3} \times (6)$

$= 2$ ✓

**Q.11** $\lim_{x\to 1} \frac{ax^2+bx+c}{cx^2+bx+a}$

$a+b+c \neq 0$

$x=1$ put

$\Rightarrow \frac{\cancel{a+b+c}}{\cancel{c+b+a}} = 1$

**Q.12** $\lim_{x\to -2} \left(\frac{\frac{1}{x}+\frac{1}{2}}{x+2}\right)$

Put $x=-2$

$\frac{0}{0}$

$= \lim_{x\to -2} \frac{\left(\frac{\cancel{2+x}}{2x}\right)}{\cancel{(x+2)}}$

$= \lim_{x\to -2} \left(\frac{1}{2x}\right)$

$= \left(\frac{1}{-4}\right)$

**Q.13** $\lim_{x\to 0} \frac{\sin ax}{bx}$

Put $x=0$

$\frac{\sin 0}{0} = \frac{0}{0}$ form

Standard Form

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} \frac{\sin ax}{bx} \times \frac{a}{a}$

$= \left(\lim_{x\to 0} \frac{\sin ax}{ax}\right) \times \frac{a}{b} = 1 \times \frac{a}{b} = \frac{a}{b}$

**Q.14** $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, $a, b \neq 0$

$= \lim_{x\to 0} \left(\frac{\sin ax}{ax}\right) \times \frac{bx}{\sin bx} \times \frac{a\cancel{x}}{b\cancel{x}}$

$= 1 \times 1 \times \frac{a}{b}$

$= \frac{a}{b}$

**Q.15** $\lim_{x\to\pi} \frac{\sin(\pi-x)}{\pi(\pi-x)}$

$\frac{\sin(\pi-\pi)}{\pi.(\pi-\pi)} = \frac{\sin 0}{\pi\times 0} = \frac{0}{0}$

By Standard Form

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \frac{1}{\pi}\times 1 = \frac{1}{\pi}$

**Q.16** $\lim_{x\to 0} \frac{\cos x}{\pi - x}$

Put $x=0$ Direct

$= \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$

**Q.17** $\lim_{x\to 0} \frac{(\cos 2x)-1}{\cos x - 1}$

Put $x=0$

$\frac{\cos 0 - 1}{\cos 0 - 1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\cos 2x = 1 - 2\sin^2 x$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$= \lim_{x\to 0} \frac{(\cancel{1} - 2\sin^2 x) - \cancel{1}}{(\cancel{1} - 2\sin^2\frac{x}{2}) - \cancel{1}} = \lim_{x\to 0} \frac{\cancel{-2}\sin^2 x}{\cancel{-2}\sin^2\frac{x}{2}}$

$= \lim_{x\to 0} \left( \frac{\sin x}{x} \cdot \frac{\frac{x}{2}\times 2}{\sin\frac{x}{2}} \right)^2$

1 1

$= (1\times 1\times 2)^2$

$= 4$

**Q.18** $\lim_{x\to 0} \frac{ax + x\cos x}{b \sin x}$   $\frac{0}{0}$ Form

Std Form
$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} \frac{\left(\frac{ax + x\cos x}{x}\right)}{\left(b \frac{\sin x}{x}\right)}$   $\frac{\sin x}{x} \to 1$

$= \lim_{x\to 0} \frac{a + \cos x}{b \times \left(\frac{\sin x}{x}\right)}$

$= \frac{a + \cos 0}{b \times 1} = \frac{a+1}{b}$

$\frac{1-\cos^2 x}{= \sin^2 x}$

**Q.19** $\lim_{x\to 0} x \sec x$

$x = 0$ put

$= 0 \times \sec 0$

$= 0 \times 1 = 0$

**Q.21** $\lim_{x\to 0} (\operatorname{cosec} x - \cot x)$

put $x = 0$
$= \operatorname{cosec} 0 - \cot 0$
$= \infty - \infty$
Indeterminate

$= \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \frac{(1-\cos x)}{\sin x} \cdot \frac{(1+\cos x)}{(1+\cos x)}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin x \cdot (1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin^{\cancel{2}} x}{\cancel{\sin x} \cdot (1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin x}{1+\cos x}$

$= \frac{\sin 0}{1+\cos 0} = \frac{0}{1+1} = \frac{0}{2} = 0$

**Q.20** $\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx}$   $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\left(\frac{\sin ax + bx}{x}\right)}{\left(\frac{ax + \sin bx}{x}\right)}$

$= \lim_{x\to 0} \frac{\frac{\sin ax}{ax} \times a + b}{a + \frac{\sin bx}{bx} \times b}$

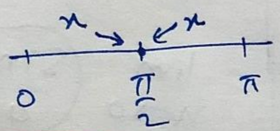
$= \frac{1 \times a + b}{a + 1 \times b} = \frac{a+b}{a+b} = 1$

Q.22 $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan 2x}{x-\frac{\pi}{2}}$

Std Form $\boxed{\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1}$

Put $x=\frac{\pi}{2}$ $\qquad \dfrac{\tan \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)}{\frac{\pi}{2}-\frac{\pi}{2}} = \dfrac{0}{0}$ Form

$\lim\limits_{x\to\frac{\pi}{2}}$




$x = \frac{\pi}{2} + h$ $\qquad h\to 0 \qquad h \approx 0$

$h = 0.000000001$

Substitute

$$= \lim_{\left(\frac{\pi}{2}+h\right)\to\frac{\pi}{2}} \left( \frac{\tan 2\left(\frac{\pi}{2}+h\right)}{\left(\cancel{\frac{\pi}{2}}+h\right) - \cancel{\frac{\pi}{2}}} \right)$$

$$= \lim_{h\to 0} \frac{\tan(\pi + 2h)}{h} \qquad \tan(\pi+\theta) = \tan\theta$$

$$= \lim_{h\to 0} \frac{\tan 2h}{h}$$

$$= \lim_{h\to 0} \frac{(\tan 2h)}{h}$$

$$= \lim_{h\to 0} 2 \cdot \underbrace{\frac{\sin 2h}{2h}}_{1} \cdot \cos 2h$$

$$= \frac{2\times 1}{\cos 0} = \frac{2}{1}$$

$$= 2$$

## Piecewise Functions

Critical point

$$f(x) = \begin{cases} x^2+3, & x \geq a \\ 2x-1, & x < a \end{cases}$$

limit at critical point

limit at non-critical point

$x = b$

Directly put

LHL — $x=a$ — RHL

$a^-$ — $a^+$

$x<a$ — $x>a$

$2a-1$ — $a^2+3$

LHL = RHL ✓

LHL ≠ RHL ✗

Here

LHL = RHL = 3

$x = 0$

$$\boxed{\lim_{x\to 0} f(x) = 3}$$

$\boxed{\text{Q.23}}$

$$f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$$

Critical point = 0

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} 3(x+1)$$

Directly put

$$= 3(1+1)$$

$$= 3\times 2 = 6 ✓$$

$$\lim_{x\to 0} f(x)$$

Critical point

Left Right

$0^-\ 0^+$

0

LHL

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} (2x+3)$$

$$= 3$$

RHL

$$\lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^-} 3(x+1)$$

$$= 3(0+1)$$

$$= 3$$

**Q.24**

$$f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$$

Critical point $= 1$

$$\lim_{x\to 1} f(x)$$

LHL

$= \lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} (x^2-1)$

$= (1)^2 - 1$

$= 1 - 1$

$= 0 = \text{LHL}$

RHL

$= \lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} (-x^2-1)$

$= -(1)^2 - 1$

$= -1 - 1$

$= -2 = \text{RHL}$

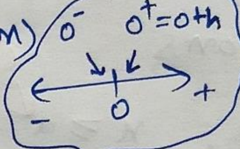
$\text{LHL} \ne \text{RHL}$

$\therefore \lim_{x\to 1} f(x)$ does not exist

**Q.25**

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x)$$




Critical point $= 0$

LHL

$= \lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} \dfrac{|x|}{x}$

$0^- = 0 - h$ $\quad h>0$ $\quad h\to 0$

$= \lim_{h\to 0} \dfrac{|-h|}{-h}$

$= \lim_{h\to 0} \dfrac{h}{-h}$

$= -1 = \text{LHL}$

RHL

$= \lim_{x\to 0^+} f(x)$

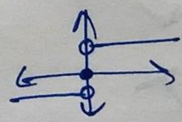
$= \lim_{x\to 0^+} \dfrac{|x|}{x}$

$0^+ = 0 + h$

$= \lim_{h\to 0} \dfrac{|0+h|}{0+h}$

$= \lim_{h\to 0} \dfrac{h}{h}$

$= 1 = \text{RHL}$



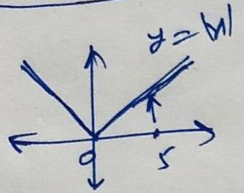
$\text{LHL} \ne \text{RHL}$

$\therefore$ Limit $\to$ Not exist

**Q.27** $f(x) = |x| - 5$

$\lim_{x \to 5} f(x)$

$|x|$ changes its nature at $x = 0$




$\lim_{x \to 5} f(x) = |5| - 5$

$= 5 - 5$

$= 0$

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**Q.28**

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$$

$a, b = ?$

Given $\underbrace{\lim_{x \to 1} f(x)}_{\text{limit}} = \underbrace{f(1)}_{\text{Exact}} = 4$

limit

neighborhood

Exact

LHL = RHL

$\boxed{LHL = RHL = 4}$

$\Rightarrow \lim_{x \to 1^-} \boxed{f(x)} = \lim_{x \to 1^+} \boxed{f(x)} = 4$

$\Rightarrow \lim_{x \to 1^-} (a + bx) = \lim_{x \to 1^+} (b - ax) = 4$

$\Rightarrow a + b = b - a = 4$

$\Rightarrow a + b = 4$ ; $a + \cancel{b} = \cancel{b} - a$

$\Rightarrow \boxed{b = 4}$ ; $\Rightarrow 2a = 0 \Rightarrow \boxed{a = 0}$

$a = 0$
$b = 4$

$\boxed{Q.29}$ $f(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$

$\lim_{x\to a_1} f(x) = \lim_{x\to a_1} (x-a_1)(x-a_2)\cdots(x-a_n)$

$= \underbrace{(a_1-a_1)}_{\to 0}(a_1-a_2)\cdots(a_1-a_n) = 0$

$\lim_{x\to a} f(x) = \lim_{x\to a} (x-a_1)(x-a_2)\cdots(x-a_n)$

$= (a-a_1)(a-a_2)\cdots(a-a_n)$

$\boxed{Q.30}$ $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$

For what value(s) of a does $\lim_{x\to a} f(x)$ exist?

$\because$ Here $|x|+1$ & $|x|-1$ are algebraic functions, therefore these functions will be continuous in their domain

$\Rightarrow$ ~~They will be~~

$\Rightarrow$ limit will exists at every point in their domain.

But Critical point $x=0$

(we have to check only on this point)

$\boxed{\lim_{x\to 0} f(x)}$

LHL RHL

$$LHL = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} (|x|+1)$$

$$= |0^-| + 1$$

$$= 0 + 1$$

$$= 1$$

$$RHL = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+} (|x|-1)$$

$$= |0| - 1$$

$$= -1$$

At $x=0$, $LHL \neq RHL$

$\therefore$ limit does not exist at $x=0$.

$\frac{0}{0}$ $\frac{\infty}{\infty}$ $0.\infty$ $\infty-\infty$

$\times$

$x \to 0-h$

$\therefore \lim_{x\to a} f(x)$ exists at every $a \in R-\{0\}$

---

~~Q.30~~ $\boxed{Q.31}$ $\lim_{x\to 1}\left(\frac{f(x)-2}{x^2-1}\right) = \pi$ $\quad \lim_{x\to 1} f(x) = ?$

$$\Rightarrow \frac{\lim_{x\to 1} f(x) - 2}{\lim_{x\to 1}(x^2) - 1} = \pi$$

$$\frac{\lim_{x\to 1} f(x) - 2}{1-1} = \pi$$

$$\frac{\lim_{x\to 1} f(x) - 2}{0} = \pi$$

$\Rightarrow$ If $\lim_{x\to 1} f(x) \neq 2$

$$\therefore \frac{\lim_{x\to 1} f(x) - 2}{\lim_{x\to 1} x^2 - 1} = \infty \neq \pi$$

$0 \neq$

$0 =$

Impossible

$\therefore \lim_{x\to 1} f(x)$ should be equal to '2'

$\frac{0}{0}$

Q.32

$$f(x) = \begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

Integers

$m, n = ?$

$\lim_{x\to 0} f(x)$ exist

$\lim_{x\to 1} f(x)$ exist

$\because \lim_{x\to 0} f(x)$ exists.

LHL = RHL

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0^-} (mx^2+n) = \lim_{x\to 0^+} (nx+m)$

$\Rightarrow m(0)^2+n = n(0)+m$

$\Rightarrow n=m$

$\lim_{x\to 1} f(x)$ exists

LHL = RHL

$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow \lim_{x\to 1^-} n(x)+m = \lim_{x\to 1^+} (nx^3+m)$

$\Rightarrow n(1)+m = n(1)+m$

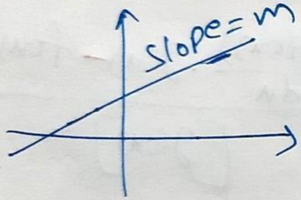
$\Rightarrow n+m = n+m$

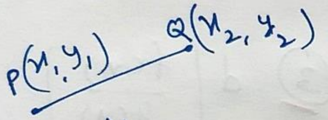
$\Rightarrow$ This is always true for every integral value of $m$ & $n$.

$\boxed{\text{Derivatives}}$ → Rate of change

Slope

Straight line $y = mx + c$

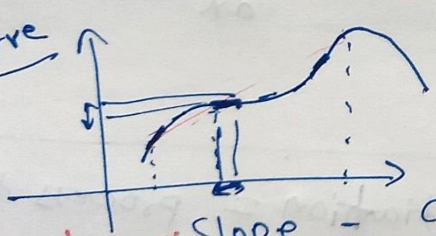



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$$\text{Slope} = m = \frac{\text{change in 'y'}}{\text{change in 'x'}} = \frac{\text{change in output}}{\text{change in input}}$$

For curve

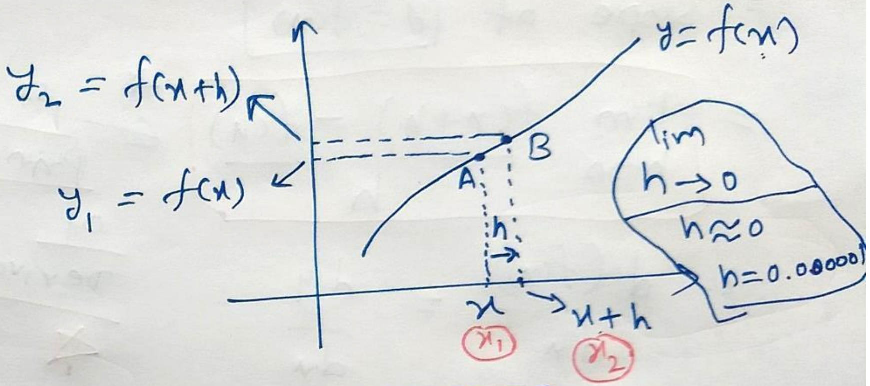


$$\text{instantaneous Slope} = \frac{\text{change in output}}{\text{very very small change in input}}$$

↕

Derivative

## Derivative of a Function $\boxed{y = f(x)}$




Derivative of $\boxed{y = f(x)}$

$$= \text{rate of change of } \boxed{y = f(x)}$$

$$= \frac{\text{change in output}}{\text{v.v. small change in input}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

☆ Derivative of $\boxed{y=f(x)}$

$= \text{Slope of } \boxed{y=f(x)}$

$= \boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}} \leftarrow$ ☆ $1^{st}$ Principle of Derivative ☆

$= \frac{d(f(x))}{dx} = \frac{dy}{dx}$

$= f'(x) = y'$

☆ Derivative (slope) at particular Point $x=a$

$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$= \left.\frac{d(f(x))}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$

$= f'(a)$

## Algebra of Derivatives

① $\frac{d[f(x)\pm g(x)]}{dx} = \frac{d f(x)}{dx} \pm \frac{d(g(x))}{dx}$

② $\frac{d[K.f(x)]}{dx} = K\frac{d(f(x))}{dx}$

③ $\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{\frac{d(f(x))}{dx}.g(x) - f(x).\frac{d(g(x))}{dx}}{[g(x)]^2}$

④ $\frac{d(f(x).g(x))}{dx} = \frac{d f(x)}{dx}.g(x) + f(x).\frac{d(g(x))}{dx}$

$\boxed{\text{Differentiation} = \text{process to find derivative}}$

☆ $(u.v)' = u'.v + u.v'$

☆ $\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

## Important Derivatives

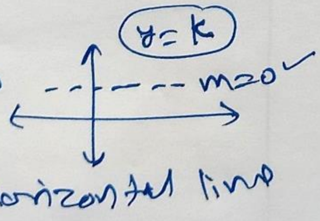
$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\text{constant})}{dx} = 0$$

$K = K.x^0$




## Derivative of $y = f(x) = x^n$ — using 1st Principle

$$\frac{d(x^n)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = {}^nC_0 . x^n . h^0 + {}^nC_1 . x^{n-1} . h^1 + \dots + {}^nC_n . x^0 . h^n$$

Binomial Theorem

$${}^nC_r = \frac{n!}{(n-r)!\, r!}$$

$${}^nC_0 = 1 = {}^nC_n$$

$${}^nC_1 = n = {}^nC_{n-1}$$

$$= x^n + n.x^{n-1}.h + \dots + h^n$$

$$\frac{d(x^n)}{dx} = \lim_{h\to 0} \frac{\left[\cancel{x^n} + n.x^{n-1}.h + \dots + h^n\right] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \left(n.x^{n-1} + \frac{n(n-1)}{2}.x^{n-2}.h + \dots + h^{n-1}\right)$$

$$= n.x^{n-1}$$

Derivative of $y = \sin x = f(x)$

Using first principle

$$\frac{d(\sin x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\boxed{\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) \cdot \cos\left(\frac{A+B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right) \cdot \cos\left(\frac{x+h+x}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right) \cdot \cos\left(x + \frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

1

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$= 1 \times \cos(x+0)$$

$$= \cos x$$

$$\boxed{\frac{d(\sin x)}{dx} = \cos x}$$

e.g. find Derivative of '$x^2 . \sin x$'

$\underbrace{x^2}_{u} \times \underbrace{\sin x}_{v}$

$(u.v)' = u'.v + u.v'$

$u = x^2$

$v = \sin x$

$u = x^2 \longrightarrow u' = \frac{d(x^2)}{dx} = 2.x^{2-1}$

$= 2.x^1$

$= 2x$

$v = \sin x \longrightarrow v' = \frac{d(\sin x)}{dx} = \cos x$

$$\frac{d(x^2 . \sin x)}{dx} = \underline{(2x).\sin x + x^2 . \cos x}$$

e.g. find derivative of '$\tan x$'

$\tan x = \frac{\sin x}{\cos x}$

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

$$\frac{d(\tan x)}{dx} = \frac{d\left(\frac{\sin x}{\cos x}\right)}{dx}$$

$$= \frac{(\sin x)' \cos x - (\sin x).(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x . \cos x - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Exercise 12.2

Q.1 Derivative of $x^2 - 2 = f(x)$ at $x = 10$.

By first principle of Derivative

$$\left.\frac{d(x^2-2)}{dx}\right|_{x=10} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h\to 0} \frac{[(10+h)^2 - 2] - [(10)^2 - 2]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{100} + h^2 + 20h - \cancel{2} - \cancel{100} + \cancel{2}}{h}$$

$$= \lim_{h\to 0} (h+20)$$

$$= 0 + 20 = 20$$

$$\frac{d(x^2-2)}{dx} = (2x^{2-1} - 0) = 2x \rightarrow 20$$

$x = 10$

Shortcut

Q.2 Derivative of $99x$ at $x=100$

$f(x) = 99x$

$$\left.\frac{d(99x)}{dx}\right|_{x=100} = \lim_{h\to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h\to 0} \frac{99(100+h) - 99\times 100}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{99\times 100} + 99\times h - \cancel{99\times 100}}{h}$$

$$= 99$$

**Q.3** Derivative of 'x' $= f(x)$ at $x=1$.

$$\left.\frac{d(x)}{dx}\right|_{x=1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(\cancel{1}+h)-\cancel{(1)}}{h}$$

$$= 1$$

**Q.4** "Derivative Using First Principle".

(i) $x^3-27 = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-27]-[x^3-27]}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^3}+h^3+3x^2h+3xh^2-\cancel{27}-\cancel{x^3}+\cancel{27}}{h}$$

$$\boxed{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$$

$(x+h)^3 =$ - — -

$$= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\left(h^2+3x^2+3xh\right)$$

$$= 0+3x^2+0 = 3x^2$$

4 (ii) $(x-1)(x-2) = f(x)$

$\Rightarrow f(x) = x^2 - 3x + 2$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(1st principle)

$$= \lim_{h\to 0} \frac{\left[(x+h)^2 - 3(x+h) + 2\right] - \left[x^2 - 3x + 2\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{3x} - 3h + \cancel{2} \overset{+\cancel{3x}-\cancel{2}}{-\cancel{x^2}}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h\to 0} (2x + h - 3) = 2x - 3 \text{ Ans.}$$

4 (iii) $\frac{1}{x^2} = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{\frac{x^2 - (x+h)^2}{x^2 \cdot (x+h)^2}}{\frac{h}{1}} \right\}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - h^2 - 2hx}{h \cdot x^2 \cdot (x+h)^2}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}(h + 2x)}{\cancel{h} \cdot (x^2) \cdot (x+h)^2}$$

$$= \frac{-2x}{x^2 \cdot x^2} = -\frac{2}{x^3}$$

4 IV $\frac{x+1}{x-1} = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h+1}{(x+h-1)} - \frac{x+1}{(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x^2-x+hx-h+x-1)-(x^2+x \overset{+hx+h}{} -x-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}-\cancel{x}+\cancel{hx}-h+\cancel{x}-\cancel{1}-\cancel{x^2}-\cancel{x}+\cancel{x}+\cancel{1}-\cancel{hx}\overset{-h}{\cancel{+h}}}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2}{(x+\underset{0}{h}-1)(x-1)}$$

$$= \frac{-2}{(x-1)^2}$$

Q.5 Prove $\boxed{f'(1) = 100 \cdot f'(0)}$ $1 = x^0$

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x^1 + 1$$

$f'(x) = ?$ $\boxed{\frac{d(x^n)}{dx} = n \cdot x^{n-1}}$ $\frac{d(\text{Const})}{dx} = 0$

$$f'(x) = \frac{100 \cdot x^{99}}{100} + \frac{99 \cdot x^{98}}{99} + \dots + \frac{2 \cdot x^1}{2} + 1 \cdot x^0 + 0$$

$$\boxed{f'(x) = x^{99} + x^{98} + \dots + x + 1}$$

$$f'(1) = \underbrace{1 + 1 + \dots + 1 + 1}_{100 \text{ times}} = 100$$

$$f'(0) = 0 + 0 + \dots + 0 + 1 = 1$$

$$f'(1) = 100 = 100 \times 1 = 100 \times f'(0)$$

Q.6 $a$ = Some fixed real no.

$$f(x) = x^n + a x^{n-1} + a^2 x^{n-2} + \dots + a^{n-1} x + a^n$$

$$\frac{d(f(x))}{dx} = \frac{d(x^n + \dots + a^n)}{dx}$$

$$= n \cdot x^{n-1} + (n-1) \cdot a \cdot x^{n-2} + (n-2) a^2 \cdot x^{n-3} + \dots + a^{n-1} \cdot 1 + 0$$

Q.7 $a, b \rightarrow$ constants

(i) $(x-a)(x-b) = f(x)$

$\Rightarrow f(x) = x^2 - ax - bx + ab$

$f(x) = x^2 - x(a+b) + ab$

$$\frac{d(f(x))}{dx} = 2x - 1 \cdot (a+b) + 0$$

$$= 2x - a - b$$

(ii) $f(x) = (ax^2+b)^2$

$f(x) = (ax^2+b)\cdot(ax^2+b)$

$\boxed{(u \cdot v)' = u'.v + u.v'}$

$$\frac{d(f(x))}{dx} = (a.2x' + 0)\cdot(ax^2+b) + (ax^2+b)\cdot(a.2x'+0)$$

$= (ax^2+b).\{2ax + 2ax\}$

$= 4ax.(ax^2+b).$

(iii) $\frac{u}{v} = \frac{x-a}{x-b} = f(x)$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{(1-0).(x-b) - (x-a).(1-0)}{(x-b)^2}$$

$$f'(x) = \frac{(x-b) - (x-a)}{(x-b)^2}$$

$$f'(x) = \frac{\cancel{x}-b-\cancel{x}+a}{(x-b)^2}$$

$$\boxed{f'(x) = \frac{a-b}{(x-b)^2}}$$

$\boxed{Q.8}$

$a =$ some constant

$x^n \to n.x^{n-1}$

$x^1 \to 1.x^{1-1} = 1.x^0 = 1\times 1 = 1$

$$f(x) = \frac{x^n - a^n}{x-a} = \frac{u}{v}$$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{(n.x^{n-1} - 0).(x-a) - (x^n - a^n).(1-0)}{(x-a)^2}$$

$$= \frac{n.x^{n-1}.(x-a) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{n.x^n - n.a.x^{n-1} - x^n + a^n}{(x-a)^2}$$

$$= \frac{(n-1)x^n - nax^{n-1} + a^n}{(x-a)^2}$$

**Q.9** (i) $2x - \frac{3}{4} = f(x)$ $\qquad \frac{d(x^n)}{dx} = n.x^{n-1}$

Derivative: $f'(x) = \frac{d\left(2x - \frac{3}{4}\right)}{dx}$

$f'(x) = 2\frac{d(x^1)}{dx} - \frac{d(3/4)}{dx} \to 0$

$= 2 \times 1.x^{1-1} - 0$

$= 2 \times x^0 = 2$

(ii) $(5x^3 + 3x - 1)(x - 1) = f(x)$

$f(x) = 5x^4 - 5x^3 + 3x - 3 - x + 1$

$f(x) = 5x^4 - 5x^3 + 2x - 2$

$f'(x) = 5.4.x^3 - 5.3.x^2 + 2 - 0$

$\boxed{f'(x) = 20x^3 - 15x^2 + 2}$

(iii) $x^{-3}(5 + 3x^1) = f(x)$

$\Rightarrow f(x) = 5x^{-3} + 3x^{-2}$ $\qquad -3 - 1 = -4$

Derivative

$\Rightarrow f'(x) = (-3).5x^{-4} + (-2)3x^{-3}$

$f'(x) = -15x^{-4} - 6x^{-3}$

(iv) $f(x) = x^5 \cdot (3 - 6x^{-9})$

$f(x) = 3x^5 - 6 \cdot x^{-4}$

Derivative

$\Rightarrow f'(x) = 5 \cdot (3x^4) - (-4) \cdot 6 \cdot x^{-5}$

$\boxed{f'(x) = 15x^4 + 24x^{-5}}$

(V) $x^{-4} \cdot (3 - 4x^{-5}) = f(x)$

$f(x) = 3x^{-4} - 4 \cdot x^{-9}$

$f'(x) = (-4) \cdot 3x^{-5} - (-9) \cdot 4 \cdot x^{-10}$

$f'(x) = -12x^{-5} + 36x^{-10}$

(VI) $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}}$$

$$f'(x) = \frac{(0) \cdot (x+1) - 2(1+0)}{(x+1)^2}$$

$$- \frac{(2x)(3x-1) - (x^2) \cdot (3 \times 1 - 0)}{(3x-1)^2}$$

$$\Rightarrow f'(x) = \left(\frac{-2}{(x+1)^2}\right) - \left(\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right)$$

$$\Rightarrow f'(x) = \left(\frac{-2}{(x+1)^2}\right) - \left(\frac{3x^2 - 2x}{(3x-1)^2}\right)$$

Q.10 Derivative of $\cos x$

(by first principle)

$f(x) = \cos x$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\Rightarrow \frac{d(\cos x)}{dx} = \lim_{h\to 0} \frac{-2\sin\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right)}{h}$$

$$\boxed{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(x+\frac{h}{2}\right) \cdot \sin\frac{h}{2}}{h}$$

Indeterminate $\rightarrow \left(\frac{0}{0}\text{ form}\right)$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \lim_{h\to 0} \left(-\sin\left(x+\frac{h}{2}\right) \cdot \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)}\right)$$

1

$$= -\sin(x+0) \times 1$$

$$= -\sin x$$

$$\boxed{\frac{d(\cos x)}{dx} = -\sin x}$$

Q.11 Find the Derivatives

(i) $f(x) = \sin x . \cos x$

$(u.v)' = u'v + u.v'$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{d(\sin x . \cos x)}{dx}$$

$$= \frac{d(\sin x)}{dx} . \cos x + \sin x . \frac{d(\cos x)}{dx}$$

$$= \cos x . \cos x + \sin x . (-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

(ii) $f(x) = \sec x = \frac{1}{\cos x} = \frac{u}{v}$

$$f'(x) = \frac{d\left(\frac{1}{\cos x}\right)}{dx}$$

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

$$= \frac{(0).\cos x - (1).(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} . \frac{1}{\cos x}$$

$$= \sec x . \tan x$$

(iii) $f(x) = 5\sec x + 4\cos x$

$$f'(x) = \frac{d(5\sec x + 4\cos x)}{dx}$$

$$= \frac{d(5\sec x)}{dx} + \frac{d(4\cos x)}{dx}$$

$$= 5.\frac{d(\sec x)}{dx} + 4.\frac{d(\cos x)}{dx}$$

By (ii) Part

$$= 5.(\sec x . \tan x) - 4\sin x$$

(iv) $f(x) = \operatorname{cosec} x = \frac{1}{\sin x} = \frac{u}{v}$

$$f'(x) = \frac{(0).\sin x - (1).(\cos x)}{\sin^2 x} \qquad \boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$= \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}.\frac{1}{\sin x} = -\cot x . \operatorname{cosec} x$$

(v) $f(x) = 3\cot x + 5\operatorname{cosec} x$

$$f(x) = \frac{3\cos x}{\sin x} + \frac{5}{\sin x}$$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{3\,d\left(\frac{\cos x}{\sin x}\right)}{dx} + \frac{5.\,d\left(\frac{1}{\sin x}\right)}{dx}$$

$$= 3.\frac{(-\sin x).\sin x - \cos x.\cos x}{\sin^2 x} + 5\left(-\cot x . \operatorname{cosec} x\right)$$

$$= -3(\sin^2 x + \cos^2 x)/\sin^2 x - 5\operatorname{cosec} x . \cot x$$

$$= -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x . \cot x$$

(vi) $f(x) = 5\sin x - 6\cos x + 7$

$\frac{d(\sin x)}{dx} = \cos x$

$\frac{d(\cos x)}{dx} = -\sin x$

$\frac{d(\text{constant})}{dx} = 0$

$f'(x) = \frac{d(5\sin x - 6\cos x + 7)}{dx}$

$= 5.(\cos x) - 6(-\sin x) + 0$

$= 5\cos x + 6\sin x$

(vii) $f(x) = 2\tan x - 7\sec x$

$f(x) = \frac{2\sin x}{\cos x} - \frac{7}{\cos x}$

$f'(x) = 2.\frac{d\left(\frac{\sin x}{\cos x}\right)}{dx} - 7.\frac{d\left(\frac{1}{\cos x}\right)}{dx}$

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

$f'(x) = 2.\frac{\cos x.\cos x - (\sin x).(-\sin x)}{\cos^2 x}$

$-7(\sec x.\tan x)$

$= 2.\frac{(\cos^2 x + \sin^2 x)}{\cos^2 x} - 7\sec x.\tan x$

$= 2\sec^2 x - 7\sec x.\tan x$

# DERIVATIVES

- First Principle of Derivatives

$$\frac{d(f(x))}{dx} = f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $(u \pm v)' = u' \pm v'$
- $(u.v)' = u'.v + u.v'$

$$(u.v.w)' = u'.v.w + u.v'.w + u.v.w'$$

- $(K\,u)' = K.u'$

constant function

- $\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

## Important Derivatives:

- $\frac{d(x^n)}{dx} = n.x^{n-1}$
- $\frac{d(\sin x)}{dx} = \cos x$
- $\frac{d(\cos x)}{dx} = -\sin x$
- $\frac{d(\tan x)}{dx} = \sec^2 x$
- $\frac{d(\sec x)}{dx} = \sec x.\tan x$
- $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$
- $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x.\cot x$

- $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d(\cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$
- $\frac{d(\cot^{-1}x)}{dx} = \frac{-1}{1+x^2}$
- $\frac{d(\sec^{-1}x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d(\operatorname{cosec}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

- $\frac{d(\log x)}{dx} = \frac{d(\log_e x)}{dx} = \frac{d(\ln(x))}{dx} = \frac{1}{x}$

Logarithmic Fn

$\boxed{e = \text{euler' No.} = 2.718\ldots}$

- $\frac{d(e^x)}{dx} = e^x$
- $\frac{d(a^x)}{dx} = a^x \cdot \log_e a$

Exponential Fn

- $\frac{d(\text{constant})}{dx} = 0$
- $\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}$
- $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$
- $\frac{d(x)}{dx} = 1$

$\frac{1}{x} = x^{-1}$

$x^n$

$\sqrt{x} = x^{\frac{1}{2}}$

**Note:** $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$\frac{d(x)}{dx} = 1$, $\frac{d(X)}{dX} = 1$, $\frac{d(y)}{dy} = 1$

$\frac{d(f(x))}{d(f(x))} = 1$  $\boxed{\frac{d(\sin x)}{dx} = \cos x}$

**Also.**

- $\frac{d\,(f(x))^n}{d(f(x))} = n.(f(x))^{n-1}$

$\sqrt[2]{x}$

- $\frac{d(\sin f(x))}{d(f(x))} = \cos(f(x))$  $\sqrt{\sin x}$

e.g. $\frac{d(\cos(\sin x))}{d(\sin x)} = -\sin(\sin x)$

$\frac{d(\cos x)}{dx} = -\sin x$

**Chain Rule:**

function in the function
$\downarrow$
Composite Function

e.g. $\sqrt{\sin x}$, $\sin(x^2)$

$\sin(\cos x)$, $\cos(\tan x)$

$f(x)$ $g(x)$ $= f(g(x))$

$(\sin x)^n = x^n$, $\sin x$

e.g. Find the derivative of $\sin(\cos x)$.

e.g. $f(x) = \sin(\cos x)$

Find $f'(x)$.

$$\frac{d(f(x))}{dx} = \frac{d(\sin(\cos x))}{dx}$$

$$= \frac{d(\sin(\cos x))}{d(\cos x)} \cdot \frac{d(\cos x)}{dx}$$

$$= \cos(\cos x) \cdot (-\sin x)$$

e.g. If $f(x) = \sin(\tan(\sqrt{x}))$

find $f'(x)$.

Ans. $$\frac{d(f(x))}{dx} = \frac{d(\sin(\tan(\sqrt{x})))}{dx}$$

$$= \frac{d(\sin(\tan\sqrt{x}))}{d(\tan\sqrt{x})} \cdot \frac{d(\tan\sqrt{x})}{d(\sqrt{x})} \cdot \frac{d(\sqrt{x})}{dx}$$

$$= \cos(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

---

Chain Rule Fast

$$\frac{d[f(g(h(x)))]}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Composite Fn.

e.g. $f(x) = \sin(\cos x)$

$f'(x) = \cos(\cos x) \cdot (-\sin x)$

e.g. $f(x) = \sin(\tan(\sqrt{x}))$

$f'(x) = \cos(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

e.g. $f(x) = \sqrt{\sin(x^2)}$

$= \left(\sin(x^2)\right)^{\frac{1}{2}}$ $\quad x^{\frac{1}{2}}$

$f'(x) = \frac{1}{\cancel{2}\sqrt{\sin(x^2)}} \cdot \cos(x^2) \cdot \cancel{2}x$

$= \frac{x \cdot \cos(x^2)}{\sqrt{\sin(x^2)}}$

## Miscellaneous Exercise 12.3

### First Principle of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Q.1** (i) $f(x) = -x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-x} - h + \cancel{x}}{h}$$

$$= -1$$

(ii) $f(x) = (-x)^{-1} = \frac{1}{(-x)} = -\frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{-1}{x+h}\right) - \left(\frac{-1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(-\frac{1}{x+h} + \frac{1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{\cancel{-x} + \cancel{x} + h}{x(x+h)}\right)}{\left(\frac{h}{1}\right)} \qquad \frac{\frac{a}{b}}{\frac{c}{b}}$$

$$= \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x(x+0)} = \frac{1}{x^2}$$

(iii) $f(x) = \sin(x+1)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+1+h) - \sin(x+1)}{h}$$

$$\boxed{\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) \cdot \cos\left(\frac{A+B}{2}\right)}$$

Standard Form.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{\cancel{x}+\cancel{1}+h-\cancel{x}-\cancel{1}}{2}\right) \cdot \cos\left(\frac{2x+2+h}{2}\right)}{h}$$

Indeterminate Form

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right) \cdot \cos\left(x+1+\frac{h}{2}\right)}{h}, \left(\frac{0}{0} \text{ Form}\right)$$

$$= \lim_{h\to 0} \boxed{\frac{\sin\left(\frac{h}{2}\right) \cdot \cos\left(x+1+\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}} = 1 \times \cos(x+1+0) = \cos(x+1)$$

$\rightarrow 1$

(iv) $f(x) = \cos\left(x - \frac{\pi}{8}\right)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(x - \frac{\pi}{8} + h\right) - \cos\left(x - \frac{\pi}{8}\right)}{h}$$

$$\boxed{\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{-2 \cdot \sin\left(\frac{2x - \frac{2\pi}{8} + h}{2}\right) \cdot \sin\left(\frac{\cancel{x} - \cancel{\frac{\pi}{8}} + h - \cancel{x} + \cancel{\frac{\pi}{8}}}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2 \cdot \sin\left(x - \frac{\pi}{8} + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-\sin\left(x - \frac{\pi}{8} + h/2\right) \cdot \sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = -\sin\left(x - \frac{\pi}{8} + 0\right) \times 1 = -\sin\left(x - \frac{\pi}{8}\right)$$

$\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \to 1$

**Q.2** $f(x) = x + a$

Derivative

$$f'(x) = \frac{d(x+a)}{dx}$$

$$= \frac{d(x)}{dx} + \frac{d(a)}{dx}$$

$$= 1 + 0$$

$$= 1$$

**Q.3** $f(x) = [p(x)+q] \cdot \left[\frac{r}{x} + s\right]$

$$f(x) = pr + ps.x + \frac{qr}{x} + q.s$$

$$f'(x) = \frac{d\left(pr + ps.x + \frac{qr}{x} + q.s\right)}{dx}$$

$$f'(x) = 0 + ps.\frac{d(x)}{dx} + qr\frac{d\left(\frac{1}{x}\right)}{dx} + 0$$

$$= ps.(1) + qr.\left(\frac{-1}{x^2}\right)$$

$$= ps - \frac{qr}{x^2}$$

$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

$$\frac{1}{x} = x^{-1}$$

$$= -1.x^{-1-1}$$

$$= -x^{-2} = \frac{-1}{x^2}$$

$\boxed{Q.4}$ $f(x) = (ax+b).(cx+d)^2$

I-method.

$f(x) = (ax+b).(cx+d).(cx+d)$

$(u.v.w)'$

$f'(x) = (a.1+0).(cx+d).(cx+d)$

$+ (ax+b).(c.1+0).(cx+d)$ ✓

$+ (ax+b).(cx+d).(c.1+0)$ ✓

$= a.(cx+d)^2 + 2.c.(ax+b).(cx+d)$

✓

$x^2 \to 2.x^{2-1} = 2x$

$x^n \to n.x^{n-1}$

II-method. (By chain Rule)

$f(x) = (ax+b).(cx+d)^2$

$(u . v)'$

$f'(x) = (a+0).(cx+d)^2.$

$+ (ax+b).\frac{d(cx+d)^2}{dx}$

$f'(x) = a.(cx+d)^2 + (ax+b).2(cx+d).(c.1+0)$

$f'(x) = a.(cx+d)^2 + 2c(ax+b).(cx+d)$

Q.5 $f(x) = \dfrac{ax+b}{cx+d} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(ax+b)'.(cx+d) - (ax+b).(cx+d)'}{(cx+d)^2}$$

$$f'(x) = \frac{(a\cdot 1+0)(cx+d) - (ax+b).(c+0)}{(cx+d)^2}$$

$$f'(x) = \frac{a\cancel{cx} + ad - a\cancel{cx} - bc}{(cx+d)^2}$$

$$f'(x) = \frac{ad - bc}{(cx+d)^2}$$

Q.6. $f(x) = \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = \dfrac{\frac{x+1}{\cancel{x}}}{\frac{x-1}{\cancel{x}}}$

$$f(x) = \frac{x+1}{x-1} = \frac{u}{v}$$

$$f'(x) = \frac{(1+0).(x-1) - (x+1).(1-0)}{(x-1)^2}$$

$$= \frac{\cancel{x}-1 - \cancel{x}-1}{(x-1)^2}$$

$$f'(x) = \frac{-2}{(x-1)^2}$$

$\boxed{Q.7}$ $f(x) = \dfrac{1}{(ax^2+bx+c)} = \dfrac{u}{v}$

$\left(\dfrac{u}{v}\right)' = \dfrac{u'.v - u.v'}{v^2}$

$f'(x) = \dfrac{(0).(ax^2+bx+c) - 1.(a.2.x + b.1 + 0)}{(ax^2+bx+c)^2}$

$x^2 \longrightarrow x^n$ $\quad \dfrac{d(x^n)}{dx} = n.x^{n-1}$

$f'(x) = \dfrac{0 - 2ax - b}{(ax^2+bx+c)^2}$

$f'(x) = \dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$ ✓

$\boxed{Q.8}$. $f(x) = \dfrac{ax+b}{px^2+qx+r} = \dfrac{u}{v}$

$f'(x) = \dfrac{(a+0).(px^2+qx+r) - (ax+b).(p2x+q+0)}{(px^2+qx+r)^2}$

$\Rightarrow f'(x) = \dfrac{a(px^2+qx+r) - (ax+b).(2px+q)}{(px^2+qx+r)^2}$

$\Rightarrow f'(x) = \dfrac{apx^2 + \cancel{aqx} + ar - 2apx^2 - \cancel{aqx} - 2bpx - bq}{(px^2+qx+r)^2}$

$\Rightarrow f'(x) = \dfrac{-apx^2 - 2bpx + ar - bq}{(px^2+qx+r)^2}$

$\boxed{Q.9}$ $f(x) = \dfrac{px^2 + qx + r}{ax + b} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(2px + q + 0).(ax+b) - (px^2+qx+r).(a+0)}{(ax+b)^2}$$

$$f'(x) = \frac{apx^2 + 2bpx - ar + bq}{(ax+b)^2}$$

$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

$\boxed{Q.10}$ $f(x) = \dfrac{a}{x^4} - \dfrac{b}{x^2} + \cos x$

$$f(x) = a.(x^{-4}) - b.(x^{-2}) + (\cos x)$$

$$f'(x) = a.\left(-4.x^{-4-1}\right) - b\left(-2.x^{-2-1}\right) + (-\sin x)$$

$$f'(x) = -\frac{4a}{x^5} + \frac{2b}{x^3} - \sin x$$

$\boxed{Q.11}$

$$f(x) = 4\sqrt{x} - 2$$

$$f(x) = 4(x)^{\frac{1}{2}} - 2$$

$$f'(x) = 4.\frac{1}{2}.x^{\frac{1}{2}-1} - 0$$

$$f'(x) = 2.x^{-\frac{1}{2}}$$

$$f'(x) = \frac{2}{x^{1/2}}$$

$$\boxed{f'(x) = \frac{2}{\sqrt{x}}}$$

**Q.12** $f(x) = (ax+b)^n$

$$f'(x) = \frac{d(ax+b)^n}{dx} \qquad \frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

Chain Rule

$$= \frac{d(ax+b)^n}{d(ax+b)} \cdot \frac{d(ax+b)}{dx}$$

$$= n \cdot (ax+b)^{n-1} \cdot (a \times 1 + 0)$$

$$= na(ax+b)^{n-1}$$

II - method. (fast) - chain Rule.

$$f(x) = (ax+b)^n$$

$$\Rightarrow f'(x) = n \cdot (ax+b)^{n-1} \cdot (a \times 1 + 0) = na(ax+b)^{n-1}$$

**Q.13** $f(x) = (ax+b)^n \cdot (cx+d)^m$

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$f'(x) = \frac{d(ax+b)^n}{dx} \cdot (cx+d)^m + (ax+b)^n \cdot \frac{d(cx+d)^m}{dx}$$

$$= n(ax+b)^{n-1} \cdot (a \times 1 + 0) \cdot (cx+d)^m$$

$$+ (ax+b)^n \cdot m \cdot (cx+d)^{m-1} \cdot (c \times 1 + 0)$$

$$= na(ax+b)^{n-1} \cdot (cx+d)^m$$

$$+ m.c\,(ax+b)^n \cdot (cx+d)^{m-1}$$

$$= (ax+b)^{n-1} \cdot (cx+d)^{m-1} \cdot \left\{ na(cx+d) + mc(ax+b) \right\}$$

$\boxed{Q.14}$ $f(x) = \sin(x+a)$

Chain Rule

$\Rightarrow f'(x) = \cos(x+a)\cdot(1+0)$

$\Rightarrow f'(x) = \cos(x+a)$ ✓

$\boxed{Q.15}$ $f(x) = \operatorname{cosec} x \cdot \cot x$

$(u\cdot v)' = u'.v + u.v'$

$$f'(x) = \frac{d(\operatorname{cosec} x\cdot \cot x)}{dx}$$

$$= \underbrace{\frac{d(\operatorname{cosec} x)}{dx}}\cdot \cot x + \operatorname{cosec} x\cdot \underbrace{\frac{d(\cot x)}{dx}}$$

$$= (-\operatorname{cosec} x\cdot \cot x)\cdot \cot x + \operatorname{cosec} x\cdot(-\operatorname{cosec}^2 x)$$

$$= -\operatorname{cosec} x\cdot \cot^2 x - \operatorname{cosec}^3 x$$

$\boxed{Q.16}$ $f(x) = \frac{\cos x}{1+\sin x} = \frac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(-\sin x)\cdot(1+\sin x) - \cos x\cdot(0+\cos x)}{(1+\sin x)^2}$$

$$f'(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$f'(x) = -\left(\frac{+\sin x + (1)}{(1+\sin x)^2}\right)$$

$$= -\frac{\cancel{(1+\sin x)}}{(1+\sin x)^{\cancel{2}}}$$

$$= -\frac{1}{(1+\sin x)}$$ ✓

Q.17 $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x} = \frac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(\cos x - \sin x).(\sin x - \cos x) - (\sin x + \cos x).(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= -\left\{\frac{\sin^2 x + \cos^2 x - \cancel{2\sin x \cos x} + \sin^2 x + \cos^2 x + \cancel{2\sin x . \cos x}}{(\sin x - \cos x)^2}\right\}$$

$$= -\frac{2}{(\sin x - \cos x)^2}$$

Q.18 $f(x) = \dfrac{\sec x - 1}{\sec x + 1} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{[\sec x . \tan x - 0].(\sec x + 1) - (\sec x - 1).[\sec x . \tan x + 0]}{(\sec x + 1)^2}$$

$$= \frac{\cancel{\sec^2 x . \tan x} + \sec x . \tan x - \cancel{\sec^2 x . \tan x} + \sec x . \tan x}{(\sec x + 1)^2}$$

$$= \frac{2 \sec x . \tan x}{(\sec x + 1)^2}$$

Q.19 $f(x) = \sin^n x$

$f(x) = (\sin x)^n$

Chain Rule

$$f'(x) = \frac{d(\sin x)^n}{dx}$$

$$= n.(\sin x)^{n-1} \times \cos x$$

$$= n.\cos x.(\sin x)^{n-1}$$

Q.20 $f(x) = \frac{a + b\sin x}{c + d\cos x}$ $\frac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{\frac{d(a+b\sin x)}{dx}.(c+d\cos x) - (a+b\sin x).\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$= \frac{(0+b\cos x).(c+d\cos x) - (a+b\sin x).[0+d(-\sin x)]}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd(1)}{(c+d\cos x)^2}$$

Q.21 $f(x) = \frac{\sin(x+a)}{\cos x}$ $\frac{u}{v}$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$f'(x) = \frac{\frac{d(\sin(x+a))}{dx} . \cos x - \sin(x+a) . \frac{d(\cos x)}{dx}}{(\cos x)^2}$$

$$= \frac{\cos(x+a).(1+0).\cos x - \sin(x+a).(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos(x+a).\cos x + \sin(x+a).\sin x}{(\cos x)^2}$$

$\cos A.\cos B + \sin A.\sin B = \cos(A-B)$

$$= \frac{\cos(\cancel{x}+a-\cancel{x})}{(\cos x)^2} = \frac{\cos a}{(\cos x)^2}$$

Q.22 $f(x) = \overset{u}{x^4} \cdot \overset{v}{(5\sin x - 3\cos x)}$

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$$f'(x) = \frac{d(x^4)}{dx} \cdot (5\sin x - 3\cos x) + x^4 \cdot \frac{d(5\sin x - 3\cos x)}{dx}$$

$$f'(x) = 4 \cdot x^3 \cdot (5\sin x - 3\cos x) + x^4 \cdot (5 \cdot \cos x + 3\sin x)$$

$$= x^3 \cdot \left\{ \begin{array}{l} 20\sin x - 12\cos x \\ + 5x\cos x + 3x\sin x \end{array} \right\}$$

Q.23 $f(x) = (x^2+1) \cdot \cos x$

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$$f'(x) = (2 \cdot x + 0) \cdot \cos x + (x^2+1) \cdot (-\sin x)$$

$$= \underline{2x\cos x} - \underline{\sin x} \cdot \underline{(x^2+1)}$$

$\boxed{Q.24}$ $f(x) = (ax^2 + \sin x) \cdot (p + q\cos x)$

$$\boxed{(u \cdot v)' = \underline{u'} \cdot \underline{v} + \underline{u} \cdot \underline{v'}}$$

$$f'(x) = (a \cdot 2x + \cos x) \cdot (p + q\cos x)$$
$$+ (ax^2 + \sin x) \cdot (0 + q(-\sin x))$$

$$f'(x) = (2ax + \cos x) \cdot (p + q\cos x)$$
$$- q \sin x \cdot (ax^2 + \sin x)$$

Q.25 $f(x) = (x+\cos x).(x-\tan x)$

$(u.v)' = u'.v + u.v'$

$$f'(x) = \frac{d(x+\cos x)}{dx}.(x-\tan x) + (x+\cos x).\frac{d(x-\tan x)}{dx}$$

$$= (1-\sin x).(x-\tan x) + (x+\cos x).(1-\sec^2 x)$$

Q.26 $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$ $\left(\frac{u}{v}\right)$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$f'(x) = \frac{(4+5\cos x).(3x+7\cos x) - (4x+5\sin x).(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{(4+5\cos x).(3x+7\cos x) - (4x+5\sin x).(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{\cancel{12x} + 28\cos x + 15x\cos x + 35\cos^2 x - \cancel{12x} + 28x\sin x - 15\sin x + 35\sin^2 x}{(3x+7\cos x)^2}$$

$$= \frac{28\cos x + 15x\cos x + 28x\sin x - 15\sin x + 35}{(3x+7\cos x)^2}$$

Q.27 $f(x) = \dfrac{x^2 \cdot \cos\frac{\pi}{4}}{\sin x}$ $\quad \dfrac{u}{v}$

$\left( \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \text{constant} \right)$

$$\frac{d f(x)}{dx} = \frac{d\left( \frac{x^2 \cdot \cos\frac{\pi}{4}}{\sin x} \right)}{dx}$$

$$= \cos\frac{\pi}{4} \cdot \frac{d\left[ \frac{x^2}{\sin x} \right]}{dx} \quad \frac{u}{v}$$

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \cos\frac{\pi}{4} \cdot \frac{(2x) \cdot \sin x - x^2 \cdot (\cos x)}{\sin^2 x}$$

$$= \cos\frac{\pi}{4} \cdot \left( \frac{2x \sin x - x^2 \cos x}{\sin^2 x} \right)$$

Q.28 $f(x) = \dfrac{x}{1 + \tan x}$ $\quad \dfrac{u}{v}$

$$f'(x) = \frac{(1) \cdot (1 + \tan x) - (x) \cdot (0 + \sec^2 x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

Q.29 $f(x) = (x + \sec x) \cdot (x - \tan x)$

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$$f'(x) = (1 + \sec x \cdot \tan x) \cdot (x - \tan x) + (x + \sec x) \cdot (1 - \sec^2 x)$$

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$f(x) = \dfrac{x}{\sin^n x} \quad \left(\dfrac{u}{v}\right)$

$\boxed{\left(\dfrac{u}{v}\right)' = \dfrac{u'.v - u.v'}{v^2}}$

$f'(x) = \dfrac{d\left(\frac{x}{\sin^n x}\right)}{dx}$

$= \dfrac{\frac{d(x)}{dx} \cdot \sin^n x - x \cdot \frac{d(\sin^n x)}{dx}}{(\sin^n x)^2}$

$\sin^n x = (\sin x)^n$

$(x^n)$ $\quad x^n \rightarrow n.x^{n-1}$

Chain Rule

$\dfrac{d(\sin x)^n}{dx} = n.(\sin x)^{n-1} . \cos(x) . 1$

$= n.\cos x.(\sin x)^{n-1}$

$= \dfrac{1 . \sin^n x - x.\left(n.\cos x.(\sin x)^{n-1}\right)}{(\sin^n x)^2}$

$= \dfrac{\sin^n x - nx.\cos x. \sin^{n-1} x}{\sin^{2n}. x}$

$= \dfrac{\cancel{\sin^{n-1} x}\,(\sin x - nx.\cos x)}{\cancel{(\sin x)^{n-1}}.(\sin x)^{n+1}}$

$= \dfrac{\sin x - nx.\cos x}{(\sin x)^{n+1}}$